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COMMENT

Comment on 'The effect of radial acceleration on the electric and magnetic fields of circular currents and rotating charges'

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Abstract

In a recent paper in this journal, Jefimenko (Jefimenko O D 2001 *J. Phys. A: Math. Gen.* **34** 6143–56) used time-dependent generalizations of the Coulomb and Biot–Savart laws to infer the existence of 'electrokinetic' and 'magnetokinetic' fields due to radial acceleration of charges in circular motion. It is shown here that for the stationary charge and current distributions discussed in Jefimenko's paper, the 'electrokinetic' and 'magnetokinetic' fields are zero. In particular, there is no 'electrokinetic' field outside an infinitely long, uniformly charged and steadily rotating hollow cylinder, and there is no electric field whatsoever outside an infinitely long cylindrical solenoid. It follows that the classical explanation of the Aharonov–Bohm effect proposed by Jefimenko is invalid.

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1. Introduction

In a recent paper [1], Jefimenko has argued that so-called 'electrokinetic' electric fields and 'magnetokinetic' magnetic-induction fields can be produced by charges in uniform circular motion, even though the charge–current distribution is everywhere constant in time. These fields are supposed to exist in addition to the usual electric and magnetic-induction fields given by the Coulomb and Biot–Savart laws for stationary sources. It is the purpose of this comment to point out (a) that Jefimenko's argument relies on an erroneous identification of the total and partial time derivatives of the current density and (b) that the 'electrokinetic' and 'magnetokinetic' fields vanish for the systems considered. It follows that an 'electrokinetic' field cannot be invoked as a classical explanation of the Aharonov–Bohm effect due to a long cylindrical solenoid and it will be shown that the field used for this purpose in [1] does not satisfy Maxwell's equations inside the solenoid.

2. Generalized Coulomb and Biot–Savart laws

The discussion in [1] starts from the following equations, which represent the electric field $\boldsymbol{E}$ and the magnetic-induction field $\boldsymbol{B}$ as volume integrals involving the charge density ρ, the current density $\boldsymbol{J}$ and their partial time derivatives $\partial\rho/\partial t$ and $\partial\boldsymbol{J}/\partial t$:

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\int\left\{\frac{[\rho]}{r^3} + \frac{1}{r^2 c}\left[\frac{\partial\rho}{\partial t}\right]\right\}\boldsymbol{r}\,\mathrm{d}V - \frac{1}{4\pi\epsilon_0 c^2}\int\frac{1}{r}\left[\frac{\partial\boldsymbol{J}}{\partial t}\right]\mathrm{d}V \tag{1}$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int\frac{[\boldsymbol{J}]}{r^3}\times\boldsymbol{r}\,\mathrm{d}V + \frac{\mu_0}{4\pi c}\int\frac{1}{r^2}\left[\frac{\partial\boldsymbol{J}}{\partial t}\right]\times\boldsymbol{r}\,\mathrm{d}V. \tag{2}$$

The notation is that used in [1]—ϵ_0 is the permittivity and μ_0 the permeability of the vacuum, c is the speed of light *in vacuo*, r is the distance from a source point in the volume element $\mathrm{d}V$ to the field point where $\boldsymbol{E}$ and $\boldsymbol{B}$ are evaluated and the square brackets indicate that the expressions inside them are to be evaluated at the retarded time $t - r/c$, where t is the time at which $\boldsymbol{E}$ and $\boldsymbol{B}$ are evaluated. Equations (1) and (2) constitute time-dependent generalizations [2–4] of the Coulomb and Biot–Savart laws

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho}{r^3}\boldsymbol{r}\,\mathrm{d}V \qquad \text{and} \qquad \boldsymbol{B} = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}}{r^3}\times\boldsymbol{r}\,\mathrm{d}V \tag{3}$$

to which they reduce if the charge–current distribution is stationary; that is, if $\partial\rho/\partial t$ and $\partial\boldsymbol{J}/\partial t$ are identically zero and, as a consequence, $[\rho] = \rho$ and $[\boldsymbol{J}] = \boldsymbol{J}$. For stationary sources, $\nabla\cdot\boldsymbol{J} = 0$ and Ampère's law

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} \tag{4}$$

is valid[1].

Jefimenko [1] refers to the contributions to $\boldsymbol{E}$ from the integrals over $[\partial\rho/\partial t]$ and $[\partial\boldsymbol{J}/\partial t]$ in equation (1) as 'electrokinetic' and to the contribution to $\boldsymbol{B}$ from the integral over $[\partial\boldsymbol{J}/\partial t]$ in equation (2) as 'magnetokinetic'. Although the terminology may suggest that these contributions to $\boldsymbol{E}$ and $\boldsymbol{B}$ are due to mere motion of the charges[2], the stronger condition that the charge–current distribution be *non-stationary* is evidently required. For example, the charge–current distribution associated with a single moving point charge is non-stationary even when the velocity is constant [6]. All the distributions considered in [1], however, are *stationary*, since uniformly charged rings, cylinders or spheres in steady rotational motion about an axis of rotational symmetry have charge and current densities that are constant in time at every space point. There are, therefore, no 'electrokinetic' or 'magnetokinetic' contributions to the corresponding fields—$\boldsymbol{E}$ and $\boldsymbol{B}$ are simply given by the Coulomb and Biot–Savart laws (3).

While noting that $\partial\rho/\partial t$ in equation (1) is zero for the systems being discussed, Jefimenko asserts that $\partial\boldsymbol{J}/\partial t$ in both equations (1) and (2), although constant in time, is not identically zero. In the case of a steadily rotating ring with uniform charge density ρ, the expression $\rho\boldsymbol{a}$ for $\partial\boldsymbol{J}/\partial t$ is obtained in equation (10) of [1]. Here $\boldsymbol{a}$ is the (centripetal) radial acceleration of the element of charge on the ring that is instantaneously located at the point where $\partial\boldsymbol{J}/\partial t$ is being evaluated. In reality, however, $\rho\boldsymbol{a}$ is the *total* time derivative of $\boldsymbol{J}$, given by

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \frac{\partial\boldsymbol{J}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{J}, \tag{5}$$

[1] Griffiths and Heald [4] use the term 'static' instead of 'stationary' to describe a charge–current distribution for which both ρ and $\boldsymbol{J}$ are time independent. These authors have also delineated more general distributions, termed 'semistatic', for which the Coulomb and Biot–Savart laws (3) still hold, although Ampère's law (4) fails.

[2] It may be of interest to note that in Maxwell's *Treatise on Electricity and Magnetism* [5], the electromagnetic energy of a system of interacting electric currents is called 'electrokinetic energy'.

and the *partial* time derivative $\partial \boldsymbol{J}/\partial t$ is identically zero[3]. In equation (5), $\boldsymbol{u}$ is the velocity $\boldsymbol{J}/\rho$ of the charge element. By using cylindrical polar coordinates and the fact that ρ is constant in the rotating ring, it may readily be verified that the second term on the right-hand side of equation (5) gives $\rho\boldsymbol{a}$. That Jefimenko has calculated the *total* time derivative of $\boldsymbol{J}$ is clear from the footnote on page 6145 of [1]. It is also clear from the derivation of equations (1) and (2) from the retarded electromagnetic potentials in the Lorentz gauge (see [4], for example) that $\partial\rho/\partial t$ and $\partial \boldsymbol{J}/\partial t$ in these equations are *partial* time derivatives at a fixed point in space.

3. Maxwell's equations

As the generalized Coulomb and Biot–Savart laws are based on Maxwell's equations, any fields $\boldsymbol{E}$ and $\boldsymbol{B}$ obtained by exact evaluation of the integrals in equations (1) and (2) must satisfy Maxwell's equations with the prescribed charge and current densities ρ and $\boldsymbol{J}$ as sources. It will now be shown that in the case of a uniformly charged and steadily rotating hollow cylinder, Jefimenko's 'electrokinetic' field $\boldsymbol{E}_{\mathrm{k}}$ does not comply with this requirement. In the limit of an infinitely long cylinder of radius b that rotates about its axis with constant angular velocity u/b and has constant surface charge density σ,

$$\boldsymbol{E}_{\mathrm{k}} = \begin{cases} \dfrac{u^2\sigma r_0}{2c^2\epsilon_0 b}\hat{\boldsymbol{r}}_0 & \text{if} \quad 0 \leqslant r_0 < b \\ \dfrac{u^2\sigma b}{2c^2\epsilon_0 r_0}\hat{\boldsymbol{r}}_0 & \text{if} \quad r_0 > b \end{cases} \tag{6}$$

where r_0 is the cylindrical–polar radial coordinate, $\hat{\boldsymbol{r}}_0$ is the outward radial unit vector and the axis of the cylinder is the z-axis. Again the notation is that of [1] except that the product of the charge density ρ and the width w of the cylindrical shell used in [1] is written here as σ; that is, the thin shell is replaced by a cylindrical surface by letting ρ tend to ∞ and w tend to zero in such a way that ρw tends to σ. The expression given in the second of equations (6) for the 'electrokinetic' field outside the cylinder $r_0 = b$ was obtained in [1] by inserting $\rho\boldsymbol{a}$ for $\partial\boldsymbol{J}/\partial t$ in equation (1) and using the value π/x^2 for the integral[4]

$$\int_0^{2\pi} \frac{\sin^2\phi}{x^2 - 2bx\cos\phi + b^2}\,\mathrm{d}\phi \tag{7}$$

in the case where $x > b$. By interchanging x and b, it can be seen that the integral (7) has the value π/b^2 when $0 < x < b$ and it is easy to verify that it has this value also when $x = 0$. The expression for $\boldsymbol{E}_{\mathrm{k}}$ inside the cylinder $r_0 = b$, given in the first of equations (6), then follows. It may be shown that $\nabla \times \boldsymbol{E}_{\mathrm{k}} = 0$ when $r_0 \neq b$. The discontinuity in $\boldsymbol{E}_{\mathrm{k}}$ on the cylinder $r_0 = b$ is removable and hence so is that of its normal component, which is $\boldsymbol{E}_{\mathrm{k}}$ itself.

The Coulomb field $\boldsymbol{E}_{\mathrm{C}}$ of the charged cylinder, which corresponds to the first integral in equation (1), is given by

$$\boldsymbol{E}_{\mathrm{C}} = \begin{cases} 0 & \text{if} \quad 0 \leqslant r_0 < b \\ \dfrac{\sigma b}{\epsilon_0 r_0}\hat{\boldsymbol{r}}_0 & \text{if} \quad r_0 > b. \end{cases} \tag{8}$$

[3] In hydrodynamics, the total time derivative is often called the derivative 'following the motion' of a fluid element and the corresponding differential operator is sometimes written as $\mathrm{D}/\mathrm{D}t$ instead of as $\mathrm{d}/\mathrm{d}t$. The total and partial derivatives give rates of change with respect to time that correspond respectively to the Lagrangian and Eulerian descriptions of fluid flow [7].

[4] The integrand has period 2π and the integral may be be evaluated over the interval $[-\pi, \pi]$ by changing the integration variable to $\tan(\phi/2)$ and using partial fractions or contour integration to integrate the resulting rational function.

This field too is radial but, in contrast to $\boldsymbol{E}_\mathrm{k}$, it changes discontinuously by $(\sigma/\epsilon_0)\hat{\boldsymbol{r}}_0$ when r_0 passes through the point b and hence it satisfies the correct boundary condition for the cylinder with surface charge density σ. Thus, the given charge on the cylinder acts as a source or a sink for the *Coulomb* field $\boldsymbol{E}_\mathrm{C}$ outside the cylinder.

Now the outward flux of $\boldsymbol{E}_\mathrm{k}$ per unit length of cylinder through any cylinder $r_0 = d$ with radius d greater than b is $u^2\pi b\sigma/(c^2\epsilon_0)$ and is independent of d. By Gauss's flux theorem, there must be charge $u^2\pi b\sigma/c^2$ per unit length of cylinder *inside* the cylinder $r_0 = b$. It follows from equations (6) that

$$\nabla \cdot \boldsymbol{E}_\mathrm{k} = \begin{cases} \dfrac{u^2\sigma}{c^2\epsilon_0 b} & \text{if} \quad 0 \leqslant r_0 < b \\ 0 & \text{if} \quad r_0 > b. \end{cases} \tag{9}$$

The first of equations (9) implies that inside the cylinder there is a constant charge density $u^2\sigma/(c^2 b)$ and hence a charge $u^2\pi b\sigma/c^2$ per unit length of cylinder that acts as a source or a sink for the field $\boldsymbol{E}_\mathrm{k}$ outside. Jefimenko's field $\boldsymbol{E}_\mathrm{k}$ is therefore inconsistent with the prescribed charge density ρ, which is non-zero only on the cylinder, and Maxwell's equation $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$ is not satisfied by the total electric field $\boldsymbol{E}_\mathrm{C} + \boldsymbol{E}_\mathrm{k}$ at any point inside the cylinder. This inconsistency confirms the result of section 2 that there is no field $\boldsymbol{E}_\mathrm{k}$ due to radial acceleration of charges on the cylinder.

4. Aharonov–Bohm effect

In section 4 of [1], an explanation, based on classical electrodynamics, of the Aharonov–Bohm effect due to an infinitely long cylindrical solenoid was proposed. For the purposes of discussion, it is convenient here to regard the solenoid as consisting of two contiguous cylinders, one with positive surface charge density $-\sigma$ at rest and the other with negative surface charge density σ rotating about its axis with constant angular velocity u/b, where σ is a negative constant and b is the radius of both cylinders. This serves as a model for the system considered in [1], in which a current of conduction electrons with negative charge density ρ flows through a closely wound solenoid against a fixed background of positive ions with charge density $-\rho$. The Coulomb fields of the two cylinders cancel but, according to [1], there is a radially inward electric field $\boldsymbol{E}_\mathrm{k}$ caused by the rotation of the negatively charged cylinder. Since this field (but no magnetic-induction field) is supposed to exist in the configuration space $r_0 > b$ of a charged particle being scattered by the solenoid, it was argued in [1] that it gives rise to the Aharonov–Bohm phase shift. There is then apparently no need to attribute a local but gauge-invariant influence to either the vector potential outside the solenoid or the particle's magnetization field inside the solenoid [8]. It has been shown here, however, that there is no 'electrokinetic' field $\boldsymbol{E}_\mathrm{k}$ accompanying the rotating cylinder and so such a field cannot be used to explain the Aharonov–Bohm effect.

It is, of course, possible for an electric field of the form given in equations (6) to exist, but this, as has been seen in section 3, would require a static charge distribution of constant negative density $u^2\sigma/(c^2 b)$ in the interior region $0 \leqslant r_0 < b$. If this charge were present, the electric field would act directly on a charged particle in the exterior region $r_0 > b$ whether the negatively charged cylinder were at rest or rotating and hence a change in the particle's wavefunction due to the electric field would occur whether the magnetic-induction field $\boldsymbol{B}$ inside the solenoid were zero or not. The Aharonov–Bohm effect, which is the additional change caused by a non-zero field $\boldsymbol{B}$ in the interior region, would still be manifest as a shift in the interference pattern of the scattered particles and would still require a non-classical interpretation.

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